Controlling the Separation of Laminar Boundary Layers in Water: Heating and Suction

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We present an analysis of the minimum surface overheat, $T_w - T_\infty$, that will delay separation of a laminar boundary layer for a prescribed adverse pressure gradient in water. The analysis is for a Falkner-Skan wedge flow corresponding to negative values of β. The energy and momentum equations are coupled through the viscosity variation with temperature. We employ a high Prandtl number approximation to obtain an asymptotic solution to these equations. The heat-transfer and viscosity variations are localized to a thin layer near the wall, well within the momentum boundary layer, and their primary effect on separation is to provide a "slip" velocity for the outer main parts of the flow, enabling the outer, shear-layer like part of the flow to sustain a more adverse pressure gradient than it could in the absence of heating. Although heating does delay separation, its effect is shown to be small for practical values of wall overheat, particularly compared to the effect of suction. For example, a suction velocity ratio of less than 0.0001 would have a comparable effect in maintaining an attached flow as an overheat of 40°F.

Introduction

We are interested in the feasibility of delaying the separation of a laminar boundary layer that is subject to an adverse pressure gradient in water. The significance of maintaining an attached boundary layer in regions of adverse pressure gradient is that transition to turbulent boundary-layer flow often occurs by laminar separation and turbulent reattachment slightly aft of the point of maximum diameter, and, as body size increases, the relative portion of body surface under turbulent flow increases. It is thus desirable to consider additional means for controlling the location of laminar separation.

Two such methods for delaying separation in the presence of an adverse pressure gradient are suction and wall heating. The effects of suction are well known. Schlichting[1] cites measurements made by Prandtl over 70 years ago that verified the powerful effect of suction on delaying separation. The effects of wall heating are not so well known. However, the calculations of Wazzan et al.[2] for a laminar water boundary layer of the Falkner-Skan type subject to an adverse pressure gradient indicate that surface heating in water could maintain an attached boundary layer, which would have separated in the absence of any temperature difference between surface and freestream. In addition, calculations of the effect of heating on separation have been performed for several flows involving regions of adverse pressure gradient: the Howarth linearly decelerating velocity distribution over a flat plate,[3] the flow over a cylinder,[3] and the flow over a sphere.[4] The movement of the separation point from its unheated value is found, by correlation, to be approximately proportional to $[Pr_\infty / Pr_w]^{1/5} - 1$. Only qualitative verification of this result has been obtained experimentally.[4]

Wazzan's[2] calculations for a Falkner-Skan $\beta = -0.198$, the maximum negative value for β for unseparated flow, also indicated that moderate surface heating in water does not increase the minimum critical Reynolds number as much as it does for flat-plate flow. This result, although not bearing on the subject of separation, strongly suggests that even if surface heating could be used to maintain an unseparated flow, the stability characteristics of this flow still would make it susceptible to transition near the pressure minimum. The situation is probably different with suction, where experience suggests that the delay of separation also leads to the delay of transition.

In this paper, the effect of wall heating on the delay of separation is analyzed for the simplest important boundary-layer flows with adverse pressure gradient: the Falkner-Skan flows with negative β and negative m, where

$$U_e \sim x^m, \quad \beta = 2m/(m+1)$$

For comparisons between suction and heating, analyses based on the Falkner-Skan flows are useful because the influence of suction on such flows has been examined carefully by Terrill[5] and Nickel.[6]

The question considered is the following: How much adverse pressure gradient will a heated laminar boundary layer in water sustain without separation? For Falkner-Skan flows, the equivalent question is: What is the largest negative value of β for an attached flow that a given wall overheat $\Delta T \equiv T_w - T_\infty$ will permit? The analysis is a systematic approximation to the boundary-layer equations, based on the properties of high Prandtl number solutions,[7] and the existence of a thermal boundary layer that is considerably thinner than the velocity boundary layer. The Prandtl number of water at 60°F is about 8. This is usually large enough for the validity of methods that are based on high Prandtl number arguments.

For water flows in the range of 40°F to 100°F, the principal departure from constant property flow is due to viscosity variation. The density, thermal conductivity, and specific heat do not vary appreciably in this range, and the most important phenomena are represented quite well by a model fluid in which ρ, k, and c_p are constant, and the variation of viscosity with temperature is preserved. There exists an analogy between wall heating and suction for a variable viscosity fluid. This follows from the compatibility condition at a wall, which

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for both suction and wall heating, and with $\mu=\mu(T)$, can be written in the form

$$\left[\rho v_w - \left(\frac{d\mu}{dT}\frac{\partial T}{\partial y}\right)_w\right]\frac{\partial u}{\partial y} = -\frac{dp}{dx} + \mu_w \frac{\partial^2 u}{\partial y^2}$$

For water, $d\mu/dT<0$, for suction $v_w<0$. Therefore, since the temperature gradient dT/dy is negative if $T_w>T_\infty$, the effect of surface heating would be qualitatively similar to that of suction. The similarity is only qualitative—suction has a more powerful and global effect on a boundary layer because of its impact on boundary-layer profiles and the growth of the boundary layer. The effect of heating is localized to a smaller region (which nevertheless is an important one for stability), and there is little effect on boundary-layer growth.

Analysis

The Falkner-Skan equations for a variable viscosity fluid are

$$(Nf'')' + ff'' + \beta[1-(f')^2] = 0 \qquad (1)$$

$$g'' + Pr_\infty f g' = 0 \qquad (2)$$

where $u/U_e = f'$, $g=(T-T_\infty)/(T_w-T_\infty)$, $N=\mu/\mu_\infty$, $Pr_\infty = \mu_\infty c_{p\infty}/k_\infty$, and primes denote differentiation with respect to η. Here $\eta = (U_e r_v^j)/(\sqrt{2\xi})$ and $\beta \equiv (2\xi/U_e)(dU_e/dx)(dx/d\xi)$, where $\xi = \rho_\infty \mu_\infty \int U_e r^{2j} dx$ and $j=0$ corresponds to plane flow and $j=1$ corresponds to axial symmetry.

The equations are to be solved subject to the boundary conditions

$$f(0)=f'(0)=0 \quad , \quad g(0)=1 \qquad (3a)$$

$$f'(\infty)=1 \quad , \quad g(\infty)=0 \qquad (3b)$$

The coupling between Eqs. (1) and (2) is through the viscosity ratio N, which is a strong function of temperature. Since we are interested in the conditions for incipient separation, we seek solutions to the foregoing set that also correspond to $f''(0)=0$. If the Prandtl number is large, then the thermal boundary layer is much thinner than the velocity boundary layer. This suggests that there is a thin region near the wall where viscosity variation is important in the momentum balance. For such a layer, the dominant terms in the force balance are shear and pressure gradient; inertial effects are negligible.

We introduce a new variable Z, such that $\eta=\epsilon Z$, so $Z=O(1)$ when $\eta=O(\epsilon)$; the dependence of this small parameter ϵ on physical quantities is determined later. Introduce a new stream function variable $X(Z)$ such that

$$f(\eta) = \epsilon^3 \beta X(Z) \qquad (4)$$

With the assumption that N is a quantity of $O(1)$, the various terms in the momentum equation are of the order indicated in the following:

$$(Nf_{\eta\eta})_\eta + ff_{\eta\eta} + \beta(1-f_\eta^2) = 0$$

$$O(1) \qquad O(\epsilon^4) \qquad O(1-\epsilon^4)$$

The determination of ϵ follows from the consideration of the energy equation; where g is of $O(1)$

$$g_{\eta\eta} + Pr_\infty f g_\eta = 0$$

$$O(1/\epsilon^2)\,O(Pr_\infty \epsilon^2 \beta)$$

It is necessary that there be a balance between conduction and transport if the boundary conditions are to be satisfied at the wall and the edge of the thermal layer. From this balance, the order of magnitude of ϵ is established: $\epsilon = (-\beta\, Pr_\infty)^{-1/4}$. The minus sign before β is chosen because β is a negative quantity. Although this quantity ϵ is not particularly small if $Pr_\infty \sim 10$ and $-\beta \sim 0.2$, there is empirical evidence from the work of Liepmann[8] and Narisimha and Vasantha[9] that, even if Pr_∞ is as small as 0.7, solutions obtained via the approximation $\epsilon \to 0$ are accurate. The reduced momentum and energy equations become

$$(NX_{ZZ})_Z + 1 = 0 \qquad (5)$$

$$g_{ZZ} - Xg_Z = 0 \qquad (6)$$

In general, these two nonlinear equations must be solved simultaneously; they are coupled through the dependence of the viscosity ratio N on g and the dependence of g on X. The appropriate boundary conditions on X are that $X(0)=X_Z(0)=0$; and the conditions on g are that $g(0)=1$, and $g(\infty)=0$. The condition that $X_{ZZ}(0)=0$ follows from the additional requirement that the relationship between β and T_w-T_∞ be such that the shear vanishes at the boundary.

Although it would be interesting to solve Eqs. (5) and (6) numerically, we can proceed to obtain quantitative estimates by introducing an additional approximation in the energy equation. Prior to stating that approximation, however, it is necessary to integrate the momentum equation.

Successive double quadrature of the momentum equation yields the formal expression

$$X_Z = -\int_0^Z \frac{Z dZ}{N(Z)} \qquad (7)$$

We integrate Eq. (7) by parts

$$\frac{dX}{dZ} = -\frac{Z^2}{2N} + \frac{1}{2}\int Z^2 \frac{d}{dZ}\left(\frac{1}{N}\right) dZ \qquad (8)$$

In the limit as $Z\to\infty$,

$$\frac{dX}{dZ} \sim \frac{-Z^2}{2} + \frac{1}{2}\int_0^\infty Z^2 \frac{d}{dZ}\left(\frac{1}{N}\right) dZ \qquad (9)$$

where we anticipate that $N\to 1$ as $Z\to\infty$.

Viscosity Variation with Temperature

It is now necessary to specify N, the viscosity ratio, as a function of g. Gazley[10] has shown that a good approximation for the kinematic viscosity of water is

$$\nu = \begin{cases} \dfrac{10^{-5}}{0.0807+0.0126\,T} & \text{for } 40°\text{F} \le T \le 100°\text{F} \\[2ex] \dfrac{10^{-5}}{-0.1946+0.0153\,T} & \text{for } 100°\text{F} \le T \le 160°\text{F} \end{cases} \qquad (10)$$

where T is in °F and ν is in ft²/sec. This suggests a viscosity-temperature model of form $\mu = 1/(a+bT)$. For such a model,

$$\frac{1}{N} = \frac{a+bT}{a+bT_\infty}$$

$$= \frac{a}{a+bT_\infty} + \frac{b[T_\infty + (T_w-T_\infty)g]}{a+bT_\infty} = 1 + \alpha\Delta T g \qquad (11)$$

where $\Delta T = T_w - T_\infty$, and $\alpha = b/(a+bT_\infty)$. For water in the temperature range between 40°F to 100°F, $\alpha \approx 0.0153$ $(°\text{F})^{-1}$. Using the relationship

$$[d(1/N)]/dZ = \alpha\Delta T(dg/dZ) \qquad (12)$$

the reduced velocity function dX/dZ is written

$$\frac{dX}{dZ} = \frac{-Z^2}{2N} + \frac{\alpha\Delta T}{2}\int_0^Z Z^2 \frac{dg}{dZ} dZ \qquad (13)$$

Approximate Solution of the Energy Equation

Heat-transfer analyses often employ crude approximations to the convective terms in the energy equation and still yield highly accurate results. Our approach is in that tradition. We assume that for purposes of solving the energy equation a suitable approximation to X is $X=-Z^3/6\langle N\rangle$, where $\langle N\rangle$ is some average reference value of N evaluated between T_∞ and T_w. (See Appendix A for a more vigorous justification of this approximation.) This is equivalent to using the constant property version of X, where the viscosity is evaluated at some reference condition in the boundary layer. Notice that

$$X\rightarrow(-Z^3/6N_w) \text{ near } Z=0 \tag{14a}$$

and

$$X\rightarrow(-Z^3/6) \text{ at large values of } Z \tag{14b}$$

This is equivalent to approximating $f(\eta)$ by $f(\eta)=-\beta\eta^3/3\langle N\rangle$ in the original variables. The resulting energy equation is

$$\frac{d^2g}{dZ^2}+\frac{1}{6\langle N\rangle}Z^3\frac{dg}{dZ}=0 \tag{15}$$

whose solution is

$$g=A\int_0^Z \exp(-Z^4/24\langle N\rangle)dZ+C \tag{16}$$

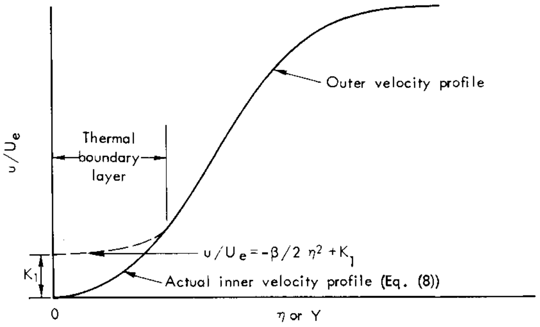

Fig. 1 Sketch of the velocity profile. Note: the outer velocity profile, if continued to the wall, would follow the dashed line and would correspond to a slip velocity, K_1 and zero shear. The inner velocity profile corrects this and brings the velocity at $y=0$ to zero while still maintaining shear zero at $y=0$. The dashed line is a parabola [see Eq. (23)] and corresponds to the effect of the surface heating on the outer velocity profile.

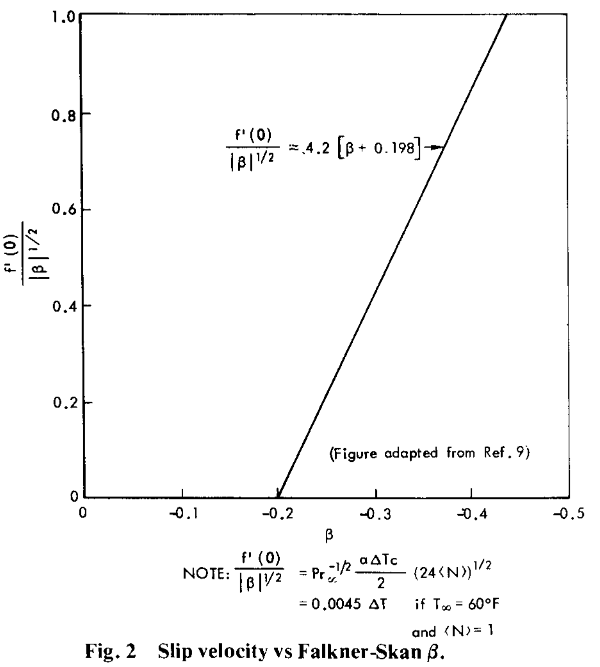

Fig. 2 Slip velocity vs Falkner-Skan β.

The boundary conditions, $g(0)=1$, $g(\infty)=0$, fix the two constants A and C so that

$$g(Z)=1-\frac{\int_0^Z \exp(-Z^4/24\langle N\rangle)dZ}{\int_0^\infty \exp(-Z^4/24\langle N\rangle)dZ} \tag{17}$$

Therefore,

$$\frac{d}{dZ}\left(\frac{1}{N}\right)=-\frac{\alpha\Delta T\cdot\exp(-Z^4/24\langle N\rangle)}{\int_0^\infty \exp(-Z^4/24\langle N\rangle)dZ} \tag{18}$$

and the asymptotic expression for dX/dZ [Eq. (9)] now can be evaluated. As $Z\rightarrow\infty$, the velocity function becomes

$$\frac{dX}{dZ}\rightarrow\frac{-Z^2}{2}-\frac{\alpha\Delta T}{2}\frac{\int_0^\infty Z^2\exp(-Z^4/24\langle N\rangle)dZ}{\int_0^\infty \exp(-Z^4/24\langle N\rangle)dZ} \tag{19}$$

Carrying out the evaluation of the preceding integrals yields

$$\frac{dX}{dZ}\rightarrow\frac{Z^2}{2}-\frac{\alpha\Delta T}{2}(24\langle N\rangle)^{-1/2}c \tag{20}$$

where $c=\Gamma(3/4)/\Gamma(1/4)$. It is important to note that the constancy of the second term on the right-hand side of Eq. (20) is not a consequence of the approximation $X=-Z^3/6\langle N\rangle$, and that because of the inequality, Eq. (A9) of Appendix A, the second term on the right-hand side of Eq. (20) must be a constant with lower and upper bounds given by

$$\frac{-\alpha\Delta T}{2}(24N_w)^{-1/2}c\leq\frac{-\alpha\Delta T}{2}B^2\int_0^\infty Z^2$$

$$\exp\left(\int_0^Z X(Z')dZ'\right)dZ\leq\frac{-\alpha\Delta T}{2}(24)^{-1/2}c \tag{21}$$

[see Eq. (A3)].

Approximate Solution of the Outer Layer

Having determined the structure of the velocity profile within the thermal boundary layer, we recast the asymptotic formula for dX/dZ in terms of the original variables, η and f

$$\frac{df}{d\eta}=\beta\epsilon^2\frac{dX}{dZ}(Z)=\beta(-\beta\, Pr_\infty)^{-1/2}\frac{dX}{dZ}(Z) \tag{22}$$

so

$$\frac{df}{d\eta}=-\frac{\beta}{2}\eta^2-\frac{\beta}{2}(-\beta\, Pr_\infty)^{-1/2}\alpha\Delta Tc(24\langle N\rangle)^{1/2} \tag{23}$$

Thus, as $\eta\rightarrow 0$, the outer flow must be of the form

$$u/U_e\rightarrow-(\beta/2)\eta^2+K_1$$

where K_1 is a positive constant, in order to match to the velocity profile at the outer edge of the thermal boundary layer.

Stewartson[11] has examined solutions of the Falkner-Skan equations that have this property. They are two-dimensional wakelike flows that have a positive slip velocity at $\eta=0$ as well as zero shear and possess values of β between -0.1988 and -0.5. Berger[12] discusses these flows and graphs the relationship between $df/d\eta$ (0) and β (see Fig. 26, p. 75, of Ref. 12). That relationship between $f_\eta(0)$ and β, in addition to

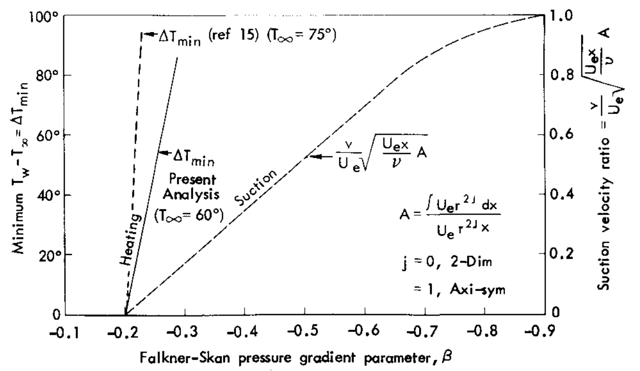

Fig. 3 Required surface heating and suction to delay laminar separation beyond $\beta = -0.1988$.

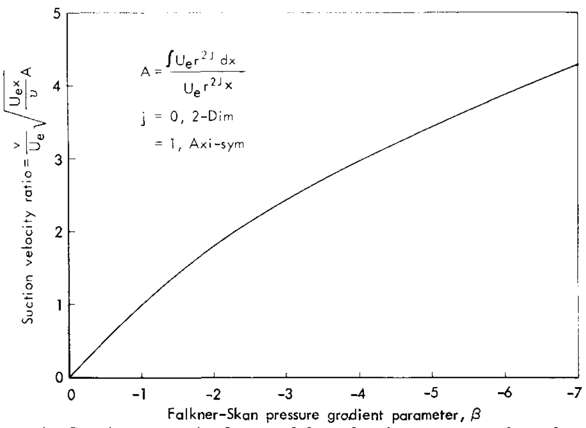

Fig. 4 Suction required to delay laminar separation beyond $\beta = 0.1988$.

Eq. (23) evaluated as $\eta \to 0$,

$$\frac{\mathrm{d}f}{\mathrm{d}\eta}(0) = -\beta(-\beta\, Pr_\infty)^{-1/2}\frac{\alpha\Delta Tc}{2}(24\langle N\rangle)^{1/2} \tag{24}$$

determines the relationship between ΔT and β that corresponds to incipient separation. (See Fig. 1 for a sketch of the outer and inner velocity profiles.)

This is replotted from Ref. 12 in the convenient form $\mathrm{d}f/\mathrm{d}\eta\,(0)/|\beta|^{1/2}$ vs β (see Fig. 2). Since

$$\frac{\mathrm{d}f}{\mathrm{d}\eta}(0)/|\beta|^{1/2} = (Pr_\infty)^{-1/2}\frac{\alpha\Delta Tc}{2}(24\langle N\rangle)^{1/2} \tag{25}$$

the ordinate is proportional to $(T_w - T_\infty)$.

Numerical Values

At $T_\infty = 60°$F, $\nu_\infty = 1.22\times10^{-5}$ ft^2/sec, $b = 0.0126\times10^5$ (ft^2/sec)$^{-1}$ (°F)$^{-1}$, and $\alpha = 0.0153$ (°F)$^{-1}$, where the density variation is neglected. If $Pr_\infty \approx 8$, then

$$(Pr_\infty)^{-1/2}(\alpha c/2)(24\langle N\rangle)^{1/2}\Delta T = 0.00453\ \Delta T\langle N\rangle^{1/2} \tag{26}$$

Results and Conclusions

The required wall overheat, $\Delta T_{\min}$, is the minimum value of the temperature difference, $T_w - T_\infty$, that will prevent separation for a particular negative value of β. Its dependence on β is shown in Fig. 3, where 60°F water is the ambient fluid. This figure assumes a reference value $\langle N\rangle$ of the viscosity ratio $N \equiv \mu/\mu_\infty$, which corresponds to unity. Since N decreases with increasing temperature, any other choice for the reference value $\langle N\rangle$ would be less than 1. From Eq. (26), it is easy to include any other choices for $\langle N\rangle$, such as $\langle N\rangle = (1 + \mu_w/\mu_\infty)/2$, which corresponds to an average viscosity ratio. The straight-line relationship between $\Delta T_{\min}$ and β is $\Delta T_{\min} \approx -1000\ [\beta + 0.1988]$. When an average reference condition is used, this straight line becomes one with positive curvature, and the resulting value of $\Delta T_{\min}$ is always greater than the $\Delta T_{\min}$ which is computed for $\langle N\rangle$ equal to one. For example, if $\Delta T_{\min}$ is 40°F, Fig. 3 indicates that $\beta = -0.24$, and the use of the average as the reference value for $\langle N\rangle$ leads to a value of β which is about -0.235. This results because $\langle N\rangle^{1/2}$ is 0.9 instead of unity.

From Fig. 3, we see that a wall overheat of 40°F is predicted to lead to a pressure gradient parameter β which is -0.24 at separation. This corresponds to a 25% increase in magnitude over the classic Falkner-Skan value of $\beta = -0.1988$. The change in β may be related to the pressure or velocity gradient using the definition of β:

$$\beta = 2\frac{1}{U_e}\frac{\mathrm{d}U_e}{\mathrm{d}x}\ \frac{\int U_e r^{2j}\mathrm{d}x}{U_e r^{2j}}$$

If separation occurs aft of the maximum diameter, then $U_e \sim U_\infty$, r is approximately a constant, and the major variation in β comes from the velocity gradient $\mathrm{d}U_e/\mathrm{d}x$. The fractional increase in the magnitude of β then corresponds roughly to the fractional increase in the magnitude of $\mathrm{d}U_e/\mathrm{d}x$. Thus, a 25% increase in the value of β corresponds to a similar increase in the permitted magnitude of the velocity gradient without separation. For the class of exact wedge flows, the freestream velocity is now $U_e \sim x^{-0.11}$ rather than $U_e \sim x^{-0.091}$, as it is for the constant property case.

In order to illustrate the effectiveness of suction compared to wall heating in delaying separation, we have shown in Fig. 3 the required suction velocity ratio, v/U_e, which will prevent separation for the same class of Falkner-Skan flows. The quantity $U_e x/\nu$ is the Reynolds number and

$$A \equiv \frac{\int U_e r^{2j}\mathrm{d}x}{U_e x r^{2j}}$$

For a boundary layer near the maximum diameter point, the quantity A is close to unity. The suction velocity ratio v_w/U_∞, which results in the same effect on separation as a 40°F wall overheat is about $0.1/\sqrt{U_\infty x/\nu}$. If the arc length Reynolds number at the maximum diameter is 10^7, then the suction velocity ratio is 0.03×10^{-3}; if the Reynolds number at the maximum diameter is 2×10^7, the value of v_w/U_∞ is 0.02×10^{-3}.

From Fig. 3, it is clear that a small amount of suction has a very powerful effect on the delay of separation. Figure 4, from Rosenhead[13] (Fig. V.10, p. 249), shows the theoretical predictions that enormously unfavorable pressure gradients can be sustained without separation. This is in accord with both intuition and experiment.[14] Wall heating, as we have shown, is much less dramatic, and for it to have an equally significant effect upon the flow unrealistically large wall overheats, $T_w - T_\infty$, are required.

The present analysis improves in accuracy as the Prandtl number increases in magnitude. However, practical values of $Pr_\infty^{1/4}$ are less than 2, suggesting that the numerical applications to water are only qualitative. Recent numerical calculations,[15] in fact, show that our analysis overestimates the effect of wall heating on β (see Fig. 3). This further reinforces our conclusion that a delay in laminar separation by heating can be obtained only if the adverse gradient is small, and that, in general, surface heating has much less impact on laminar separation than even a small amount of suction.

Appendix A: Approximation $X \sim -Z^3$

A more rigorous justification of the use of the approximation $X \sim -Z^3$ in the energy equation can be given as

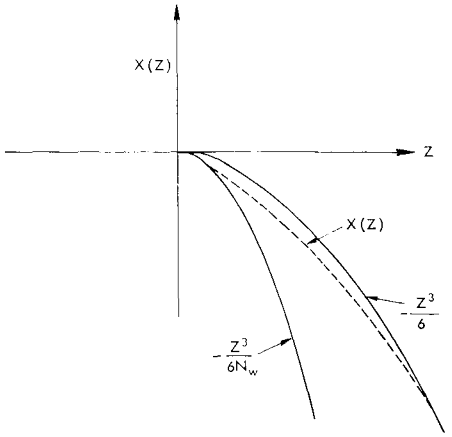

Fig. A1 Sketch of approximate solution of energy equation.

follows. The matching rule used in Eqs. (22) and (23) is formally

$$\lim_{\eta\to 0} \frac{df(\eta)}{d\eta} = \lim_{Z\to\infty} \frac{d}{d\eta}[\epsilon^3\beta X(Z)] = \epsilon^2\beta \lim_{Z\to\infty} \frac{dX}{dZ} \quad (A1)$$

where $\lim_{Z\to\infty} dX/dZ$ is determined from Eq. (13). Equation (13), which is exact, requires that dg/dZ be known from the solution of the energy equation, Eq. (6). This latter equation formally can be solved exactly for arbitrary $X(Z)$,

$$\frac{dg}{dZ} = -B^2\exp\left(\int_0^Z X(Z)dZ\right) \quad (A2)$$

where B is an arbitrary real constant determined by the boundary conditions, and the constant in Eq. (A2) is written as $-B^2$ to emphasize the fact that $dg/dZ<0$ for all Z. Substituting this result into Eq. (13), we obtain

$$\frac{dX}{dZ} = -\frac{Z^2}{2N} - \frac{\alpha B^2 \Delta T}{2}\int_0^Z Z^2\exp\left(\int_0^Z X(Z')dZ'\right)dZ \quad (A3)$$

From the definition of N and the result just deduced above $dg/dZ<0$ for all Z, we note that $dN/dZ>0$ for all Z. But $N=N_w$ at $Z=0$ and $N=1$ at $Z=\infty$, so

$$N_w \leq N(Z) \leq 1 \quad (A4)$$

This result implies

$$-\frac{1}{N_w}\int_0^Z ZdZ \leq -\int_0^Z \frac{ZdZ}{N(Z)} \leq -\int_0^Z ZdZ \quad (A5)$$

or, using Eq. (7),

$$-(Z^2/2N_w) \leq X_Z \leq -(Z^2/2) \text{ for all } Z \quad (A6)$$

Since $N>0$, it follows from the first integral of the momentum equation, Eq. (5), that

$$X_{ZZ}<0 \text{ for all } Z \quad (A7)$$

Equation (7) shows also that

$$X_Z<0 \text{ for all } Z \quad (A8)$$

Recalling that X approaches limits as $Z\to 0$ and $Z\to\infty$ given by Eqs. (14), we assert that the consequence of Eqs. (A6, A7, and A8) is that the curve $X(Z)$ must lie between the curves $-Z^3/6N_w$ and $-Z^3/6$, as shown in Fig. A1, coinciding with the first of these near the origin, approaching the latter as $Z\to\infty$, and having the general shape indicated. In particular, the curve for $X(Z)$ can never cross into the regions of the plane outside the two solid curves for this would violate either the left-or right-hand side of inequality, Eq. (A6), or make X a double-valued function of Z. It then follows that

$$-\frac{Z^4}{24N_w} \leq \int_0^Z X(Z')dZ' \leq \frac{-Z^4}{24} \quad (A9)$$

This is the result used earlier to obtain the bounds on the second term in Eq. (13) quoted earlier, Eq. (21).

Since for the cases of practical interest here, N_w is close to 1 (e.g., for water with $\Delta T=40°F$, $N_w=0.8$), we see from Eq. (A3) that it is reasonable to make the approximation

$$\int_0^Z X(Z')dZ' = -\frac{Z^4}{24\langle N\rangle} \quad (A10)$$

where $\langle N\rangle$ is some average reference value of N evaluated for some T lying between T_∞ and T_w. This is the approximation made in the main text. However, we are seeking a lower bound for the temperature differences, $T_w - T_\infty$, which will have a desired effect, the magnitude of which effect is determined by the value of the right-hand side of Eq. (A3) for that ΔT. This is obtained from Eq. (A9) by assigning to the integral appearing on the left-hand side of Eq. (A10) its upper limit, namely $-Z^4/24$. In fact, the specific numerical estimates given in the paper involve the use of this latter value, since they are based on $\langle N\rangle = 1$.

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